Math 246A Lecture 14 Notes

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1 The Argument Principle

1.1 The argument principle

Last time, we proved a number of properties of the winding number $n(\gamma, \alpha)$.

Corollary 1.1. *Let D be a disc, $\gamma \subseteq D$, γ piecewise C^1 and closed, and let $\alpha \in D \setminus \gamma$. Then*

$$n(\gamma, \alpha) f(\alpha) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - \alpha} \, dz$$

Proof. Observe that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - \alpha} \, dz - n(\gamma, \alpha) f(\alpha) = \frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(\alpha)}{z - \alpha} \, dz = 0$$

because this integral is zero on all rectangles R with $\partial R \subseteq D$. □

Theorem 1.1 (argument principle)**.** *Let D, γ, f be as above. Let $a \in \mathbb{C} \setminus f(\gamma)$. Then*

$$n(f(\gamma), a) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} \, dz = \sum_{\substack{z \in D \\ f(z) = a}} n(\gamma, z)$$

Proof. Let $w = f(z)$. By change of variables,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} \, dz = \frac{1}{2\pi i} \int_{f(\gamma)} \frac{1}{w - a} \, dw = n(f(\gamma), a).$$

For the second equality, without loss of generality, $a = 0$. Let $z_1, z_2, \dots, z_N$ be zeros of f in $\{z : |z - z_0| < (R + 1)/2\}$, where $\gamma \subseteq \{z : |z - z_0| < r < R\}$, with multiplicities n_j. Let

$$g(z) = \frac{f(z)}{\prod_{j=1}^N (z - z_j)^{n_j}}$$

Then

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{j=1}^{N} \frac{n_j}{z - z_j},$$

so

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz + \sum_{j=1}^{N} n_j n(\gamma, z_j).$$

It remains to show that the integral on the right hand side is zero. But g'/g is holomorphic in $\{z : |z - z_0| < (R+1)/2\}$, so this integral is zero. $\square$

1.2 Local zeros of analytic functions

Corollary 1.2. *Let f be analytic in $\{z : |z - z_0| < R\}$, and suppose that $f(z_0) = w_0$ with $f(z) - w_0$ having a zero of order N at z_0. That is,*

$$f(z) = w_0 + a_N(z - z_0)^N + \cdots$$

with $a_N \neq 0$. Then there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ then there exists $\delta > 0$ such that if $|w - w_0| < \delta$, then $f(z) = w$ has N solutions on $|z - z_0| < \varepsilon$.

Proof. Let $\gamma = \{z : |z - z_0| = \varepsilon\}$, and let $\delta = \int_\gamma |f(z) - w_0|$. Then

$$n(f(\gamma), w) = n(f(\gamma), w_0) = N.$$

The left hand side is the number of points (counting multiplicity) in $\{z : |z - z_0| < \varepsilon\}$ with $f(z) = w$. $\square$

Corollary 1.3 (open mapping theorem)**.** *Let $f \in H(\Omega)$ be nonconstant for a domain Ω. Then $f(\Omega)$ is open. If f is 1 to 1, then $f'(z) \neq 0$ for all z and f^{-1} is analytic.*

Proof. For the second part, since f is 1 to 1, the winding number must be 1. So f is locally 1 to 1, so $f'(z) \neq 0$. The limit $\lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)}$ exists and equals $1/f'(z_0)$. $\square$